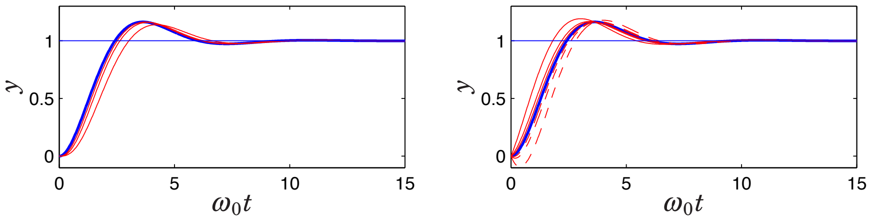

Figure 1.37 Effect of additional poles and zeros. The left figure shows step responses of a system with additional fast poles and the right plot shows responses of systems with additional fast zeros in the left (full lines) right (dashed lines) half plane. The fast singularities are 2, 5 and 10 times larger than ω_0.

will behave like a second order system with the same poles. The behavior is modified if there are other poles close to the dominant poles. Poles and zeros that are close are called *dipoles*. They influence tracking of slow signals but have little effect on the step response. Poles and zeros to the left of the dominant poles have little influence on the transient response if they are sufficiently far away from the dominant poles.

The following example shows that fast stable poles and fast zeros have little influence on the step response.

Example 1.9—Effect of Fast Poles and Zeros

Figure 1.37 illustrates the effect of additional fast poles and zeros. The left figure shows the step responses of systems that have dominant poles with damping ratio $\zeta = 0.5$ at a distance ω_0 from the origin and extra poles at $2\omega_0, 5\omega_0$ and $10\omega_0$. The figure shows that poles 5 to 10 times faster than the dominant poles have small influence on the step response.

The right figure shows the effect of an extra zero at $\pm 2\omega_0, \pm 5\omega_0$ and $\pm 10\omega_0$. Curves corresponding to right half plane zeros are shown in dashed lines, notice that these responses have negative overshoot. The effect of the zero is small if they have distances longer than 5 to $10\omega_0$. □

Relations between the requirements and the pole-zero configuration will now be discussed for simple systems. †

t: Dipoler flyttat till LeftOver

First Order Systems

Consider a system with the transfer function

$$G(s) = \frac{a}{s+a} = \frac{1}{1+sT} \tag{1.76}$$

where T is the time constant of the system. This transfer function also describes the response of the process output to set-point changes for a system with error feedback, where the loop transfer function is

$$G_l(s) = \frac{G(s)}{1-G(s)} = \frac{a}{s} = \frac{1}{sT}$$

Table 1.3 Properties of the transfer function $G_{yy_{sp}} = a/(s+a)$.

Property	Value
Rise time	$T_r = 1/a = T$
Settling time (2%)	$T_s = 4/a = 4T$
Average residence time	$T_{ar} = T = 1/a$
Overshoot	$o = 0$
Error coefficients	$e_0 = 0,\ e_1 = 1/a = T$
Bandwidth	$\omega_b = a$
Sensitivities	$M_s = M_t = 1$
Gain margin	$g_m = \infty$
Phase margin	$\varphi_m = 90°$
Gain crossover frequency	$\omega_{gc} = a$

The step response $h(t)$ and the impulse responses $g(t)$ of the system $G(s)$ are

$$h(t) = 1 - e^{-at} = 1 - e^{-t/T}, \qquad g(t) = ae^{-at} = \frac{1}{T}e^{-t/T}$$

Simple calculations give the properties of the step response shown in Table 1.3. The step and impulse responses are monotone. The velocity constant e_1 is also equal to the time constant T. This means that there will be a constant tracking error of $e_1 v = v_0 T$ when the input signal is a ramp $r = v_0 t$. The Nyquist curve of the loop transfer function is the negative imaginary axis, which implies that the phase margin is 90°. This system has a gain crossover frequency $\omega_{gc} = a$.

Second Oder System without Zeros

Consider a second order system with the transfer function

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \tag{1.77}$$

The system $G(s)$ has two poles, they are complex if $\zeta < 1$ and real if $\zeta \geq 1$. The step response of the system is

$$h(t) = \begin{cases} 1 - \dfrac{e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi), & \phi = \arccos\zeta \quad \text{for } |\zeta| < 1 \\ 1 - (1 + \omega_0 t)e^{-\omega_0 t} & \text{for } \zeta = 1 \\ 1 - \dfrac{e^{-\zeta\omega_0 t}}{\sqrt{\zeta^2 - 1}}\sinh(\omega_d t + \phi), & \phi = \arccos\zeta \quad \text{for } |\zeta| > 1 \end{cases}$$

where $\omega_d = \omega_0\sqrt{|1-\zeta^2|}$ and $\phi = \arccos\zeta$. When $\zeta < 1$ the step response is a damped oscillation, with frequency $\omega_d = \omega_0\sqrt{1-\zeta^2}$. The step response is enclosed by the envelopes

$$e^{-\zeta\omega_0 t} \leq h(t) \leq 1 - e^{-\zeta\omega_0 t}$$

Table 1.4 Properties of the response to reference values of a second order system.

Property	Value
Rise time	$T_r = e^{\phi/\tan\phi}/\omega_0$
Settling time (2%)	$T_s \approx 4/(\zeta\omega_0)$
Average residence time	$T_{ar} = 2\zeta/\omega_0$
Peak time	$T_p \approx \pi/\omega_d$
Overshoot	$o = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$
Error coefficients	$e_0 = 0,\ e_1 = 2\zeta/\omega_0$
Damped natural frequency	$\omega_d = \omega_0\sqrt{1-\zeta^2}$
Bandwidth	$\omega_b = \omega_0\sqrt{1-2\zeta^2+\sqrt{(1-2\zeta^2)^2+1}}$
Maximum sensitivity	$M_s = \sqrt{\dfrac{8\zeta^2+1+(4\zeta^2+1)\sqrt{8\zeta^2+1}}{8\zeta^2+1+(4\zeta^2-1)\sqrt{8\zeta^2+1}}}$
Sensitivity frequency	$\omega_{ms} = \dfrac{1+\sqrt{8\zeta^2+1}}{2}\omega_0$
Resonance peak	$M_t = M_p = \begin{cases} 1/(2\zeta\sqrt{1-\zeta^2}) & \text{if } \zeta \le \sqrt{2}/2 \\ 1 & \text{if } \zeta > \sqrt{2}/2 \end{cases}$
Resonance frequency	$\omega_{mp} = \begin{cases} \omega_0\sqrt{1-2\zeta^2} & \text{if } \zeta \le \sqrt{2}/2 \\ 1 & \text{if } \zeta > \sqrt{2}/2 \end{cases}$
Gain margin	$g_m = \infty$
Phase margin	$\varphi_m = 90^\circ - \arctan\omega_{gc}/(2\zeta\omega_0)$
Gain crossover frequency	$\omega_{gc} = \omega_0\sqrt{\sqrt{4\zeta^4+1}-2\zeta^2}$
Sensitivity crossover	$\omega_{sc} = \omega_0/\sqrt{2}$

The step response settles like a first order system with time constant $T = 1/(\zeta\omega_0)$. The 2% settling time is $T_s \approx 1/(4\zeta\omega_0)$. Step responses for different values of ζ are shown in Figure XXX in Chapter XXX. The maximum of the step response occurs approximately at $T_p \approx \pi/\omega_d$, i.e. half a period of the oscillation. The overshoot depends on the damping. The largest overshoot is 100% for $\zeta = 0$. Properties of the step response are summarized in Table 1.4.

The system (1.77) can be interpreted as a feedback system with the loop transfer function

$$G_l(s) = \frac{G(s)}{1-G(s)} = \frac{\omega_0^2}{s(s+2\zeta\omega_0)}.$$

The properties of the system (1.77) can then also be related to the properties of the loop transfer function $G_l(s)$. Specific relations are given in Table 1.4.

Second Order Systems with Zero

Consider the system with the transfer function

$$G(s) = \frac{\omega_0^2}{a}\cdot\frac{s+a}{s^2+2\zeta\omega_0 s+\omega_0^2}, \qquad (1.78)$$

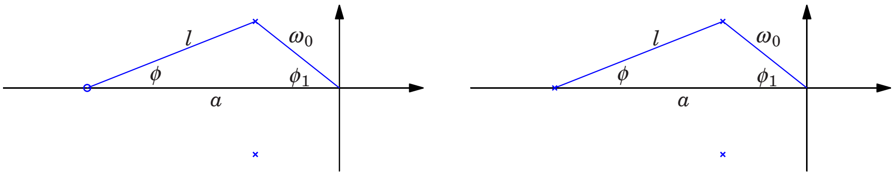

Figure 1.38 Pole-zero pattern for the transfer functions (1.78) left and (1.80) right.

† **t**: Ändra nollställe till pol i högra figuren

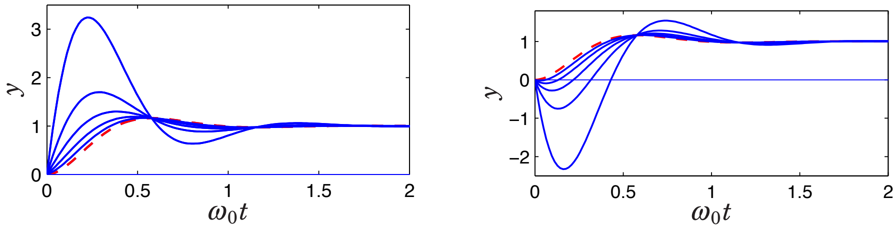

Figure 1.39 Step responses for the transfer function (1.78). The parameters are $\zeta = 0.5$ and $a = 0.2, 0.5, 1, 2, 5, \infty$ in the left figure and $a = -0.2, -0.5, -1, -2, -5, \infty$ in the right figure.

which has been normalized so that the zero frequency gain is one. The pole-zero diagram of the system is shown in Figure 1.38.

For $\zeta < 1$ the step response is

$$y(t) = 1 - \frac{l}{a\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin(\omega_d t + \phi + \phi_1), \tag{1.79}$$

where $\omega_d = \omega_0\sqrt{1-\zeta^2}$, $\phi = \arccos\zeta$, and $\phi_1 = \arccos(a - \zeta\omega_0)/l_1$. The parameters ϕ, ϕ_1, a and l have nice geometric interpretations as is shown in Figure 1.38.

Step responses are shown in Figure 1.39. The zero has a large effect on the response if it is close to the origin but little influence if it is far away ($a > 5\omega_0$). Notice that the angle ϕ_1 is negative if the zero is in the right half plane. The response then has negative overshoot.

Third Order System

Consider a system with the transfer function

$$G(s) = \frac{\omega_0^2 a}{(s^2 + 2\zeta\omega_0 s + \omega_0^2)(s+a)} \tag{1.80}$$

When $\zeta < 1$ the step response is

$$y(t) = 1 - \frac{a}{l\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin(\omega_d t + \phi - \phi_1) - \frac{\omega_0^2}{l^2} e^{-at},$$

where $\omega_d = \omega_0\sqrt{1-\zeta^2}$, $\phi = \arccos\zeta$ and $\phi_1 = \arccos(a-\zeta\omega_0)/l$. The parameters ϕ, ϕ_1, a and l have nice geometric interpretations as is shown in Figure 1.38

Standard Forms for the Complementary Sensitivity Function

Summarizing we find that the following transfer functions are reasonable forms for the complementary sensitivity function for low order closed loop systems

$$T_1 = \frac{b_c}{s+a_c},\ T_2 = \frac{\omega_c(\beta s+\omega_c)}{s^2+2\zeta\omega_c s+\omega_c^2},\ T_3 = \frac{\alpha\omega_c^2(\beta s+\omega_c)}{(s^2+2\zeta\omega_c s+\omega_c^2)(s+\alpha\omega_c)}. \quad (1.81)$$

The parameter ω_c is a scale factor that determines performance, such as bandwidth, response speed, etc. Parameters α, β, and ζ determine the shape of the transfer functions. Relative damping ζ is less than one for oscillatory systems, and larger than one when the system has real poles. The parameter α and β have a significant influence if $\alpha < 1$ and $\beta > 1$. Negative values of β give dynamics with inverse response. Decreasing α makes the response slower and reduces the overshoot. Increasing β makes the response and increases the overshoot. All transfer function (1.81) give zero steady state error for step inputs, choosing $\beta = 2\zeta$ in T_2 and $\beta = 1+2\alpha\zeta$ in T_3 gives systems that have zero steady state errors for ramp inputs.

Properties such as settling time, bandwidth and robustness are given by the complementary sensitivity function T_k, disturbance attenuation and measurement noise also requires information about the controller or the process.

The transfer function T_2 with $\zeta = \sqrt{2}/2$ and $\beta = 0$ and T_3 with $\zeta = 0.5$, $\alpha = 1$ and $\beta = 0$ have maximally flat frequency response meaning several derivatives of the gain at $\omega = 0$ are zero. The transfer function T_2 with $\zeta = \sqrt{2}/2$ and T_3 with $\zeta = 0.44$, $\alpha = 0.60$ and $\beta = 0$ minimize ITAE. The transfer functions T_3 with $\zeta = 1.6$ and T_2 with $\zeta = 0.42$ and $\alpha = 0.22$ minimized ITAE for zero steady state error for ramp response.

The transfer function

$$T_4(s) = \frac{a_n\omega_0^n}{s^n + a_1\omega_0 s^{n-1} + a_2\omega_0^2 s^{n-2} + \cdots + a_n\omega_0^n} \quad (1.82)$$

has step responses with zero steady state error. The coefficients a_k can be chosen to obtain many different properties. Coefficients that minimize ITAE are given in Table 1.5. The transfer function

$$T_5(s) = \frac{a_{n-1}\omega_0^{n-1}s + a_n\omega_0^n}{s^n + a_1\omega_0 s^{n-1} + a_2\omega_0^2 s^{n-2} + \cdots + a_n\omega_0^n} \quad (1.83)$$

give steady state responses to steps and ramps with zero steady state errors and minimal ITAE for step responses if the coefficients a_k are chosen as in Table 1.5.

1.9 Summary

† Requirements for a control system should pay attention to load disturbances, measurement noise, process uncertainty and set-point response. For systems

Tore: Lär noga och kolla speciellt tabellen